Saddlepoint approximations to the probability of ruin in finite time for the compound Poisson risk process perturbed by diffusion (Complement)

Riccardo Gatto Benjamin Baumgartner

2014

Introduction

This document serves as a complement to

[GB] Gatto, R. and Baumgartner, B. (2016). "Saddlepoint approximations to the probability of ruin in finite time for the compound Poisson risk process perturbed by diffusion". *Methodology and Computing in Applied Probability* **18**(1), pp. 217–235.

The joint or double Laplace transform of the time of ruin T and the initial capital Y_0 is given in GB (Theorem 2.1). A proof using the Laplace transform of the Gerber–Shiu function is given in GB (Appendix) and a complete proof, which does not require the Gerber–Shiu function, is the following.

Alternative proof of Theorem 2.1

Proof of Theorem 2.1. Within any small time interval of length $h > 0$, either zero, one or more than one jumps (or claims) occur with the corresponding probabilities $1 - \lambda h + \mathrm{o}(h)$, $\lambda h + \mathrm{o}(h)$ and $\mathrm{o}(h)$, as $h \downarrow 0$. Thus, from the independence and the stationarity of the increments of $\{Y_t\}_{t \geq 0}$, we obtain

$$\begin{aligned} f_\alpha(x) &= (1-\lambda h)\,\mathbb{E}\mathbb{E}\left[\mathrm{e}^{\alpha(T+h)}\mathbb{1}(T<\infty) \,\middle|\, Y_0 = x + ch + \sigma W_h\right] \\ &\quad + \lambda h\,\mathbb{E}\mathbb{E}\left[\mathrm{e}^{\alpha(T+h)}\mathbb{1}(T<\infty) \,\middle|\, Y_0 = x + ch + \sigma W_h - X_1\right] + \mathrm{o}(h) \\ &= (1-\lambda h)\,\mathrm{e}^{\alpha h}\,\mathbb{E}\left[f_\alpha\left(x + ch + \sigma W_h\right)\right] + \lambda h\,\mathrm{e}^{\alpha h}\,\mathbb{E}\left[f_\alpha\left(x + ch + \sigma W_h - X_1\right)\right] + \mathrm{o}(h), \end{aligned} \tag{1}$$

as $h \downarrow 0$ and for any $\alpha \in \mathbb{R}$ such that $f_\alpha(x) < \infty$. The first expectation in (1) can be written as

$$\begin{aligned} \mathbb{E}\left[f_\alpha\left(x + ch + \sigma W_h\right)\right] &= f_\alpha(x) + \mathbb{E}\left[\sum_{k=1}^{2} \frac{f_\alpha^{(k)}(x)}{k!}\left(ch + \sigma W_h\right)^k + \mathrm{o}\left(\{ch + \sigma W_h\}^2\right)\right] \\ &= f_\alpha(x) + \sum_{k=1}^{2}\left(\frac{f_\alpha^{(k)}(x)}{k!}\sum_{i=0}^{k}\binom{k}{i}(ch)^{k-i}\,\sigma^i\,\mathbb{E}\left[W_h^i\right]\right) + \mathbb{E}\left[\mathrm{o}\left(W_h^2\right)\right] \\ &= f_\alpha(x) + chf_\alpha'(x) + \tfrac{1}{2}\sigma^2 h f_\alpha''(x) + \mathrm{o}(h). \end{aligned} \tag{2}$$

Similarly, the second expectation in (1) can be written as

$$\begin{aligned}
\mathbb{E}\big[f_\alpha\big(x+ch+\sigma W_h - X_1\big)\big] &= \mathbb{E}\big[f_\alpha(x-X_1)\big] + \mathbb{E}\left[\sum_{k=1}^{2}\frac{f_\alpha^{(k)}(x-X_1)}{k!}\big(ch+\sigma W_h\big)^k + \mathrm{o}\big(\{ch+\sigma W_h\}^2\big)\right] \\
&= \mathbb{E}\big[f_\alpha(x-X_1)\big] + \sum_{k=1}^{2}\left(\frac{\mathbb{E}\big[f_\alpha^{(k)}(x-X_1)\big]}{k!}\sum_{i=0}^{k}\binom{k}{i}(ch)^{k-i}\,\sigma^i\,\mathbb{E}\big[W_h^i\big]\right) + \mathbb{E}\big[\mathrm{o}\big(W_h^2\big)\big] \\
&= \mathbb{E}\big[f_\alpha(x-X_1)\big] + ch\mathbb{E}\big[f'_\alpha(x-X_1)\big] + \tfrac{1}{2}\sigma^2 h\mathbb{E}\big[f''_\alpha(x-X_1)\big] + \mathrm{o}(h). \qquad (3)
\end{aligned}$$

As indicated by a Referee, the existence of f''_α, required in the Taylor expansions in (2) and (3), is established by Feng (2011, Lemma C.1), because the function f_α is a special case of the more general functional of T given in GB (Equation 15). Replacing the expectations in (1) with their respective expansions (2) and (3), dividing both sides by $\mathrm{e}^{\alpha h}h(1-\lambda h)$ and rearranging terms results in

$$\begin{aligned}
0 &= \frac{1}{h}\left(1-\frac{\mathrm{e}^{-\alpha h}}{1-\lambda h}\right)f_\alpha(x) + cf'_\alpha(x) + \tfrac{1}{2}\sigma^2 f''_\alpha(x) \\
&\quad + \frac{1}{1-\lambda h}\Big(\mathbb{E}\big[f_\alpha(x-X_1)\big] + ch\mathbb{E}\big[f'_\alpha(x-X_1)\big] + \tfrac{1}{2}\sigma^2 h\mathbb{E}\big[f''_\alpha(x-X_1)\big]\Big) + \mathrm{o}(1).
\end{aligned}$$

Let $g(h) = \mathrm{e}^{-\alpha h}(1-\lambda h)^{-1}$, then by the rule of de l'Hospital the coefficient of $f_\alpha(x)$ converges to $\lim_{h\downarrow 0}\big(g(0)-g(h)\big)/h = -g'(0) = \alpha-\lambda$. Thus, by letting $h\downarrow 0$, we obtain the integro-differential equation

$$0 = \tfrac{1}{2}\sigma^2 f''_\alpha(x) + cf'_\alpha(x) + (\alpha-\lambda)f_\alpha(x) + \lambda\mathbb{E}\big[f_\alpha(x-X_1)\big]$$

or, equivalently,

$$0 = \tfrac{1}{2}\sigma^2 f''_\alpha(x) + cf'_\alpha(x) + (\alpha-\lambda)f_\alpha(x) + \lambda\int_0^x f_\alpha(x-\xi)\,\mathrm{d}F_X(\xi) + \lambda[1-F_X(x)]. \qquad (4)$$

In the next step, both sides of (4) are multiplied by $\mathrm{e}^{\beta x}$ and integrated from 0 to ∞. This corresponds to taking Laplace transforms with reversed sign of the argument β, thus $\widehat{f'_\alpha}(\beta) = -\beta\hat{f}_\alpha(\beta) - f_\alpha(0)$ and $\widehat{f''_\alpha}(\beta) = \beta^2\hat{f}_\alpha(\beta) + \beta f_\alpha(0) - f'_\alpha(0)$, for any $\beta\in\mathbb{R}$ such that $\hat{f}_\alpha(\beta)<\infty$, where $\hat{g}(u) = \int_0^\infty \mathrm{e}^{ux}g(x)\,\mathrm{d}x$, for a generic function g. As a consequence,

$$\begin{aligned}
0 &= \tfrac{1}{2}\sigma^2\big[\beta^2\hat{f}_\alpha(\beta) + \beta f_\alpha(0) - f'_\alpha(0)\big] + c\big[-\beta\hat{f}_\alpha(\beta) - f_\alpha(0)\big] \\
&\quad + (\alpha-\lambda)\hat{f}_\alpha(\beta) + \lambda\hat{f}_\alpha(\beta)M_X(\beta) - \tfrac{\lambda}{\beta}\big[1-M_X(\beta)\big],
\end{aligned}$$

which, when solved for $\hat{f}_\alpha(\beta)$, which is the left-hand side of (5), leads to

$$\begin{aligned}
\hat{f}_\alpha(\beta) &= \frac{\big(c-\tfrac{1}{2}\sigma^2\beta\big)f_\alpha(0) + \tfrac{1}{2}\sigma^2 f'_\alpha(0) - \tfrac{\lambda}{\beta}\big(M_X(\beta)-1\big)}{\tfrac{1}{2}\sigma^2\beta - c\beta + \alpha + \lambda\big(M_X(\beta)-1\big)} \\
&= \frac{\big(c-\tfrac{1}{2}\sigma^2\beta\big)f_\alpha(0) + \tfrac{1}{2}\sigma^2 f'_\alpha(0) - \tfrac{\kappa(\beta)}{\beta} + \tfrac{1}{2}\sigma^2\beta - c}{\kappa(\beta)+\alpha}.
\end{aligned}$$

Note that $\hat{f}_\alpha(\beta)$ exists for all $\beta<0$, if $\alpha<0$. In particular, it exists for $\beta=\nu(\alpha)<0$ with $\alpha<0$. In this case the above denominator vanishes and therefore $\nu(\alpha)$ is a common root of both the denominator and numerator above, otherwise the existence of $\hat{f}_\alpha(\nu(\alpha))$ would be contradicted. Because of that, setting the numerator equal to 0 and substituting $\nu(\alpha)$ for β yields

$$\tfrac{1}{2}\sigma^2 f'_\alpha(0) = -\big(c-\tfrac{1}{2}\sigma^2\nu(\alpha)\big)\,f_\alpha(0) - \frac{\alpha}{\nu(\alpha)} - \tfrac{1}{2}\sigma^2\nu(\alpha) + c,$$

and hence

$$\hat{f}_\alpha(\beta) = \frac{\frac{1}{2}\sigma^2\big(\beta - \nu(\alpha)\big)\big(1 - f_\alpha(0)\big) - \frac{\kappa(\beta)}{\beta} - \frac{\alpha}{\nu(\alpha)}}{\kappa(\beta) + \alpha}.$$

Without initial reserve, i. e. for $x = 0$, ruin occurs almost surely and $T = 0$ a. s. because the regularity of the Wiener process implies that the risk process crosses the null level infinitely often over any arbitrarily small time interval containing the origin. Hence $f_\alpha(0) = 1$ and, as a consequence, GB (Equation 12), i. e.

$$\hat{f}_\alpha(\beta) = -\frac{\frac{\alpha}{\nu(\alpha)} + \frac{\kappa(\beta)}{\beta}}{\alpha + \kappa(\beta)} \tag{5}$$

holds for all $\alpha, \beta < 0$.

From the fact that $D = \{(\alpha, \beta) \in \mathbb{R}^2 \colon \alpha \leq \mathring{\alpha},\ \beta < \tilde{\nu}(\alpha)\}$ is a connected subset of $\mathbb{R}^2$ and the right-hand side of (5) is an analytical function for all $\alpha, \beta \in D$, follows that the double Laplace transform formula (5) holds over the entire set D. □

A helpful picture of the domain D is given by Figure 1.

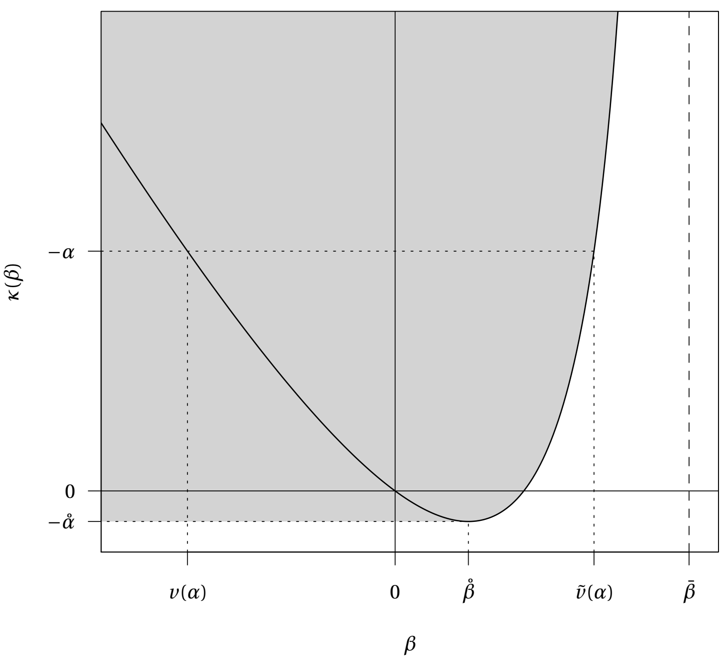

Figure 1: A representation of the domain D of the double Laplace transform $\hat{f}_\alpha(\beta)$