Saddlepoint approximations to the probability of ruin in finite time for the compound Poisson risk process perturbed by diffusion (Complement)

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Introduction

This document serves as a complement to

> [GB] Gatto, R. and Baumgartner, B. (2016). "Saddlepoint approximations to the probability of ruin in finite time for the compound Poisson risk process perturbed by diffusion". *Methodology and Computing in Applied Probability* **18**(1), pp. 217–235.

The joint or double Laplace transform of the time of ruin T and the initial capital Y_0 is given in GB (Theorem 2.1). A proof using the Laplace transform of the Gerber–Shiu function is given in GB (Appendix) and a complete proof, which does not require the Gerber–Shiu function, is the following.

Alternative proof of Theorem 2.1

Proof of Theorem 2.1. Within any small time interval of length $h > 0$, either zero, one or more than one jumps (or claims) occur with the corresponding probabilities $1 - \lambda h + \mathrm{o}(h)$, $\lambda h + \mathrm{o}(h)$ and $\mathrm{o}(h)$, as $h \downarrow 0$. Thus, from the independence and the stationarity of the increments of $\{Y_t\}_{t \geq 0}$, we obtain

$$\begin{aligned} f_\alpha(x) &= (1-\lambda h)\,\mathbb{E}\mathbb{E}\big[\mathrm{e}^{\alpha(T+h)}\mathbb{1}(T<\infty) \,\big|\, Y_0 = x + ch + \sigma W_h\big] \\ &\quad + \lambda h\,\mathbb{E}\mathbb{E}\big[\mathrm{e}^{\alpha(T+h)}\mathbb{1}(T<\infty) \,\big|\, Y_0 = x + ch + \sigma W_h - X_1\big] + \mathrm{o}(h) \\ &= (1-\lambda h)\,\mathrm{e}^{\alpha h}\,\mathbb{E}\big[f_\alpha\big(x+ch+\sigma W_h\big)\big] + \lambda h\,\mathrm{e}^{\alpha h}\,\mathbb{E}\big[f_\alpha\big(x+ch+\sigma W_h - X_1\big)\big] + \mathrm{o}(h), \end{aligned} \tag{1}$$

as $h \downarrow 0$ and for any $\alpha \in \mathbb{R}$ such that $f_\alpha(x) < \infty$. The first expectation in (1) can be written as

$$\begin{aligned} \mathbb{E}\big[f_\alpha\big(x+ch+\sigma W_h\big)\big] &= f_\alpha(x) + \mathbb{E}\left[\sum_{k=1}^{2}\frac{f_\alpha^{(k)}(x)}{k!}\big(ch+\sigma W_h\big)^k + \mathrm{o}\left(\{ch+\sigma W_h\}^2\right)\right] \\ &= f_\alpha(x) + \sum_{k=1}^{2}\left(\frac{f_\alpha^{(k)}(x)}{k!}\sum_{i=0}^{k}\binom{k}{i}(ch)^{k-i}\,\sigma^i\,\mathbb{E}\big[W_h^i\big]\right) + \mathbb{E}\left[\mathrm{o}\left(W_h^2\right)\right] \\ &= f_\alpha(x) + chf_\alpha'(x) + \tfrac{1}{2}\sigma^2 h f_\alpha''(x) + \mathrm{o}(h). \end{aligned} \tag{2}$$

Similarly, the second expectation in (1) can be written as

$$\begin{aligned}\mathbb{E}\big[f_\alpha\big(x+ch+\sigma W_h-X_1\big)\big] &= \mathbb{E}\big[f_\alpha(x-X_1)\big]+\mathbb{E}\left[\sum_{k=1}^{2}\frac{f_\alpha^{(k)}(x-X_1)}{k!}\big(ch+\sigma W_h\big)^k+\mathrm{o}\big(\{ch+\sigma W_h\}^2\big)\right]\\ &= \mathbb{E}\big[f_\alpha(x-X_1)\big]+\sum_{k=1}^{2}\left(\frac{\mathbb{E}\big[f_\alpha^{(k)}(x-X_1)\big]}{k!}\sum_{i=0}^{k}\binom{k}{i}(ch)^{k-i}\,\sigma^i\,\mathbb{E}\big[W_h^i\big]\right)+\mathbb{E}\left[\mathrm{o}\big(W_h^2\big)\right]\\ &= \mathbb{E}\big[f_\alpha(x-X_1)\big]+ch\mathbb{E}\big[f_\alpha'(x-X_1)\big]+\tfrac{1}{2}\sigma^2h\mathbb{E}\big[f_\alpha''(x-X_1)\big]+\mathrm{o}(h). \qquad (3)\end{aligned}$$

As indicated by a Referee, the existence of f_α'', required in the Taylor expansions in (2) and (3), is established by Feng (2011, Lemma C.1), because the function f_α is a special case of the more general functional of T given in GB (Equation 15). Replacing the expectations in (1) with their respective expansions (2) and (3), dividing both sides by $\mathrm{e}^{\alpha h}h(1-\lambda h)$ and rearranging terms results in

$$\begin{aligned}0 &= \frac{1}{h}\left(1-\frac{\mathrm{e}^{-\alpha h}}{1-\lambda h}\right)f_\alpha(x)+cf_\alpha'(x)+\tfrac{1}{2}\sigma^2f_\alpha''(x)\\ &\quad+\frac{1}{1-\lambda h}\Big(\mathbb{E}\big[f_\alpha(x-X_1)\big]+ch\mathbb{E}\big[f_\alpha'(x-X_1)\big]+\tfrac{1}{2}\sigma^2h\mathbb{E}\big[f_\alpha''(x-X_1)\big]\Big)+\mathrm{o}(1).\end{aligned}$$

Let $g(h)=\mathrm{e}^{-\alpha h}(1-\lambda h)^{-1}$, then by the rule of de l'Hospital the coefficient of $f_\alpha(x)$ converges to $\lim_{h\downarrow 0}\big(g(0)-g(h)\big)/h=-g'(0)=\alpha-\lambda$. Thus, by letting $h\downarrow 0$, we obtain the integro-differential equation

$$0=\tfrac{1}{2}\sigma^2f_\alpha''(x)+cf_\alpha'(x)+(\alpha-\lambda)f_\alpha(x)+\lambda\mathbb{E}\big[f_\alpha(x-X_1)\big]$$

or, equivalently,

$$0=\tfrac{1}{2}\sigma^2f_\alpha''(x)+cf_\alpha'(x)+(\alpha-\lambda)f_\alpha(x)+\lambda\int_0^x f_\alpha(x-\xi)\,\mathrm{d}F_X(\xi)+\lambda[1-F_X(x)]. \qquad (4)$$

In the next step, both sides of (4) are multiplied by $\mathrm{e}^{\beta x}$ and integrated from 0 to ∞. This corresponds to taking Laplace transforms with reversed sign of the argument β, thus $\widehat{f_\alpha'}(\beta)=-\beta\hat{f}_\alpha(\beta)-f_\alpha(0)$ and $\widehat{f_\alpha''}(\beta)=\beta^2\hat{f}_\alpha(\beta)+\beta f_\alpha(0)-f_\alpha'(0)$, for any $\beta\in\mathbb{R}$ such that $\hat{f}_\alpha(\beta)<\infty$, where $\hat{g}(u)=\int_0^\infty\mathrm{e}^{ux}g(x)\,\mathrm{d}x$, for a generic function g. As a consequence,

$$\begin{aligned}0 &= \tfrac{1}{2}\sigma^2\big[\beta^2\hat{f}_\alpha(\beta)+\beta f_\alpha(0)-f_\alpha'(0)\big]+c\big[-\beta\hat{f}_\alpha(\beta)-f_\alpha(0)\big]\\ &\quad+(\alpha-\lambda)\hat{f}_\alpha(\beta)+\lambda\hat{f}_\alpha(\beta)M_X(\beta)-\tfrac{\lambda}{\beta}\big[1-M_X(\beta)\big],\end{aligned}$$

which, when solved for $\hat{f}_\alpha(\beta)$, which is the left-hand side of (5), leads to

$$\begin{aligned}\hat{f}_\alpha(\beta) &= \frac{\big(c-\tfrac{1}{2}\sigma^2\beta\big)f_\alpha(0)+\tfrac{1}{2}\sigma^2f_\alpha'(0)-\tfrac{\lambda}{\beta}\big(M_X(\beta)-1\big)}{\tfrac{1}{2}\sigma^2\beta-c\beta+\alpha+\lambda\big(M_X(\beta)-1\big)}\\ &= \frac{\big(c-\tfrac{1}{2}\sigma^2\beta\big)f_\alpha(0)+\tfrac{1}{2}\sigma^2f_\alpha'(0)-\tfrac{\kappa(\beta)}{\beta}+\tfrac{1}{2}\sigma^2\beta-c}{\kappa(\beta)+\alpha}.\end{aligned}$$

Note that $\hat{f}_\alpha(\beta)$ exists for all $\beta<0$, if $\alpha<0$. In particular, it exists for $\beta=\nu(\alpha)<0$ with $\alpha<0$. In this case the above denominator vanishes and therefore $\nu(\alpha)$ is a common root of both the denominator and numerator above, otherwise the existence of $\hat{f}_\alpha(\nu(\alpha))$ would be contradicted. Because of that, setting the numerator equal to 0 and substituting $\nu(\alpha)$ for β yields

$$\tfrac{1}{2}\sigma^2f_\alpha'(0)=-\big(c-\tfrac{1}{2}\sigma^2\nu(\alpha)\big)\,f_\alpha(0)-\frac{\alpha}{\nu(\alpha)}-\tfrac{1}{2}\sigma^2\nu(\alpha)+c,$$

and hence

$$\hat{f}_\alpha(\beta) = \frac{\frac{1}{2}\sigma^2\bigl(\beta - \upsilon(\alpha)\bigr)\bigl(1 - f_\alpha(0)\bigr) - \frac{\kappa(\beta)}{\beta} - \frac{\alpha}{\upsilon(\alpha)}}{\kappa(\beta) + \alpha}.$$

Without initial reserve, i. e. for $x = 0$, ruin occurs almost surely and $T = 0$ a. s. because the regularity of the Wiener process implies that the risk process crosses the null level infinitely often over any arbitrarily small time interval containing the origin. Hence $f_\alpha(0) = 1$ and, as a consequence, GB (Equation 12), i. e.

$$\hat{f}_\alpha(\beta) = -\frac{\frac{\alpha}{\upsilon(\alpha)} + \frac{\kappa(\beta)}{\beta}}{\alpha + \kappa(\beta)} \tag{5}$$

holds for all $\alpha, \beta < 0$.

From the fact that $D = \{(\alpha,\beta) \in \mathbb{R}^2 \colon \alpha \le \mathring{\alpha},\ \beta < \tilde{\upsilon}(\alpha)\}$ is a connected subset of $\mathbb{R}^2$ and the right-hand side of (5) is an analytical function for all $\alpha, \beta \in D$, follows that the double Laplace transform formula (5) holds over the entire set D. $\square$

A helpful picture of the domain D is given by Figure 1.

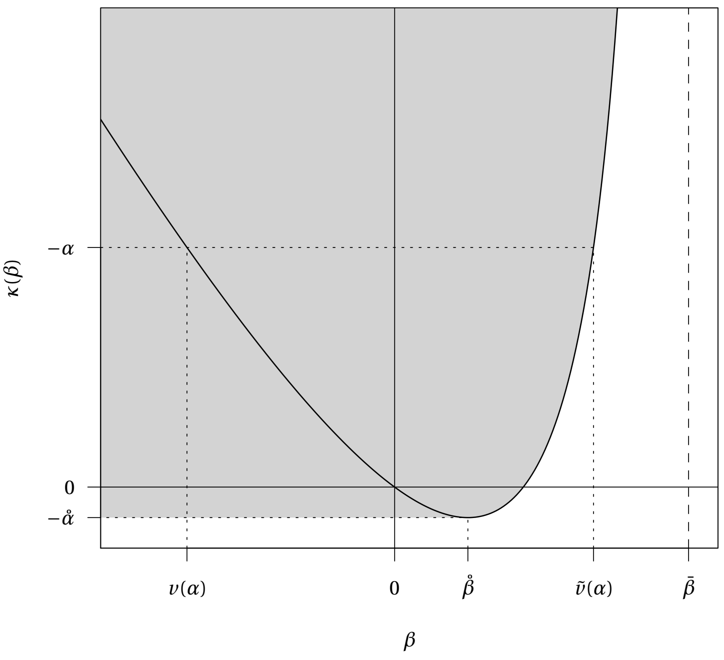

Figure 1: A representation of the domain D of the double Laplace transform $\hat{f}_\alpha(\beta)$